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Asymptotic Matching by the Symbolic Manipulator MACSYMA

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We consider delegating the tedious labor of calculating higher-order terms in singular perturbation expansions to a computer by using the algebraic manipulation system MACSYMA. In particular, the method of matched asymptotic expansions for two model singular perturbation problems have been studied in detail. The first model resembles the boundary-layer equation, with a small parameter multiplying the highest derivative. In this case, both the outer and inner expansions are simple power series in ε. The second model is a turning-point problem. The asymptotic expansions that resemble the series in low Reynolds number flow past a sphere are mixtures of powers and logarithms in ε, and Fraenkel's restricted matching principle is needed. We show that MACSYMA has successfully performed the higher order matching in these two problems.

1. INTRODUCTION

The method of matched asymptotic expansions has been a powerful tool in solving singular perturbation problems of layer type. The series approximations often stop at the first few terms. Extending the series by hand is hardly feasible due to the mounting labor that is involved in higher order calculation. In regular perturbations, the routine labor has been successfully delegated to a computer. The problem is ordinarily chosen so simple that the basic approximation is known in closed form and recurrence relations can be found for successive terms. The FORTRAN language is then used to do the arithmetic operations. In some cases (Van Dyke [1, 2]), dozens or even hundreds of terms can be found. With a sufficient number of terms, one will be able to analyze the structure of the series and to recast so as to improve its utility.

It is conceivable that a similar approach can be taken in singular perturbation problems. On the other hand, the method of matched asymptotic expansions involves mainly algebraic oprations. A symbolic manipulator will be more appropriate than a simple arithmetic language, and MACSYMA is chosen for this task. MACSYMA has also been used to extend regular perturbation series (Ander-

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son and Geer [3], Hue and Tenti [4]). But with the mounting storage requirement for the symbolic manipulation system, MACSYMA has not been used as often as FORTRAN in series expansion for regular perturbation problems.

MACSYMA is a collection of programs embedded in a LISP interpreter. It is devoted to the manipulation of algebraic expressions that includes differentiation, integration, taking limits, solving equations, expanding functions in power series or Laurent series, etc. It is also a programming language in itself, with a syntax similar to ALGOL. The version that we used during this research runs under the Incompatible Time-Sharing System (ITS), on the MACSYMA Consortium (MC) PDP-10 computer at MIT.

We consider extending the singular perturbation series in two model problems with different kinds of asymptotic expansions. In Section 2, the method of matched asymptotic expansions is stated and the detailed formulations of the two models are discussed. Our experience in programming in MACSYMA is described in Section 3. Finally, in the fourth section, the results and the future use of MACSYMA in asymptotic matching are discussed.

2. Mathematical Formulation

In this section, we shall first briefly summarize the method of matched asymptotic expansions and then describe the formulations of the two models.

2.1. *Method of Matched Asymptotic Expansions*

Matched asymptotic expansions is a technique for solving an initial- or boundary-value problem for a function $F(x, \varepsilon)$ over a range of x in which no single asymptotic series for small ε can approximate F uniformly. Usually two asymptotic series are needed to describe the function F in two different regions in the domain of interest. At least one of the two problems describing the series will not be well posed since the initial or boundary conditions outside that region are lost. One will have to use "matching" in the overlap domain in order to recover those missing data. The prototype of a singular perturbation problem is Prandtl's boundary-layer theory.

Matching not only provides information on the undetermined constants in finding the asymptotic expansions, but also suggests (or checks) the appropriate asymptotic sequence in the inner (or outer) series.

Van Dyke [5] proposed a simple-to-use matching principle (the asymptotic matching principle) which says that the m-term inner expansion of (the n-term outer expansion) is equal to the n-term outer expansion of (the m-term inner expansion), where m and n can be any integers. For convenience in programming and avoiding the confusion in counting terms, we introduce the following notations. Let $g(y, \varepsilon)$ be the inner expansion with inner variable y, and $f(x, \varepsilon)$ be the outer expansion with outer variable x. They are related in the following ways: $g(y, \varepsilon) = f(x, \varepsilon)$

and $y = x/\varepsilon$. With $O_{m,n}$ denoting the operator[1] that truncates the asymptotic series for fixed outer variable x as $\varepsilon \to 0$ up to and including the $\varepsilon^m \ln^n \varepsilon$ term, and $I_{i,j}$ denoting the operator that truncates the asymptotic series for fixed inner variable y as $\varepsilon \to 0$ up to and including the $\varepsilon^i \ln^j \varepsilon$ term, the asymptotic matching principle can be written as

$$O_{m,n} I_{i,j}\, g(y, \varepsilon) = I_{i,j} O_{m,n} f(x, \varepsilon). \tag{1}$$

For the case of pure power series expansions, we will omit the second subscript and the matching principle (1) becomes

$$O_m I_i\, g(y, \varepsilon) = I_i O_m f(x, \varepsilon). \tag{2}$$

Fraenkel [6] warns that the asymptotic matching principle (1) may fail in the case that the asymptotic sequences are mixtures of powers and logarithms in ε unless n and j are restricted to zero, i.e.,

$$O_{m,0} I_{i,0}\, g(y, \varepsilon) = I_{i,0} O_{m,0} f(x, \varepsilon). \tag{3}$$

A less restricted matching principle is suggested by Lo [7], namely, in the case that the asymptotic sequences are known, the matching can be done in the following ways[2]:

$$[O_{m,0} I_{i,j}\, g(y, \varepsilon)]_y = [I_{i,j} O_{m,0} f(x, \varepsilon)]_y \tag{4a}$$

or

$$[O_{m,n} I_{i,0}\, g(y, \varepsilon)]_x = [I_{i,0} O_{m,n} f(x, \varepsilon)]_x. \tag{4b}$$

We will compare the efficiencies between using (3) and (4) in the last section.

2.2. *Formulation of Model* I

Our first model resembles the boundary-layer equation, in that a small parameter multiplies the highest derivative. The function $f(x, \varepsilon)$ satisfies the following differential equation:

$$\varepsilon \frac{d^2 f}{dx^2} + \frac{df}{dx} + f = 0 \tag{5}$$

and the boundary conditions $f(0, \varepsilon) = A$ and $f(1, \varepsilon) = B$. This differential equation can be solved in closed form but we seek a series solution. Setting $\varepsilon = 0$ reduces the differential equation to first order, so one of the boundary conditions must be abandoned. The differential equation (5) suggests $f(x, \varepsilon)$ to be the outer function, x to be

[1] We restrict our consideration to situation when expansions proceed in integral powers of ε or mixtures of integral powers and logarithms of ε.

[2] The subscript y (or x) indicates that the final equation in the matching is done in the inner (or outer) variable.

the outer variable, and $f(1, \varepsilon) = B$ to be the only boundary condition being enforced. $f(x, \varepsilon)$ has a simple asymptotic expansion in ε, namely

$$f(x, \varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m f_m(x). \tag{6}$$

The solutions for the f_m are

$$\begin{aligned} m &= 0 \qquad f_0(x) = Be^{1-x} \\ m &\geq 1 \qquad f_m(x) = e^{-x} \int_1^x dx'\, e^{x'}(-f''_{m-1}(x')) \end{aligned} \tag{7}$$

The approximation of $f(x, \varepsilon)$ in (6) breaks down within the "boundary layer" where $x = O(\varepsilon)$. Introduce an inner coordinate y and an inner function $f(y, \varepsilon)$ which satisfy $y = x/\varepsilon$ and $g(y, \varepsilon) = f(x, \varepsilon)$. $g(y, \varepsilon)$ is governed by the transformed problem

$$\frac{d^2 g}{dy^2} + \frac{dg}{dy} + \varepsilon g = 0, \tag{8}$$

with the inner boundary condition $g(0, \varepsilon) = A$. The inner function also has a simple asymptotic expansion in ε, namely

$$g(y, \varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m g_m(y). \tag{9}$$

The solutions for the g_m are

$$\begin{aligned} m &= 0 \qquad g_0(y) = A + C_0(1 - e^{-y}) \\ m &\geq 1 \qquad g_m(y) = C_m(1 - e^{-y}) \\ &\qquad\qquad - \int_0^y dy'\, g_{m-1}(y') + e^{-y} \int_0^y dy'\, e^{y'} g_{m-1}(y') \end{aligned} \tag{10}$$

The C_m are constants that will be determined by the asymptotic matching principle. The f_m and g_m in (7) and (10) can all be found in closed form in terms of elementary functions. We will discuss how to handle an integrand that cannot be integrated out in closed form in the following section.

2.3. *Formulation of Model* II

Our second model can be described by the following differential equation

$$\varepsilon^2 \frac{d^2 f}{dx^2} + \frac{2x}{1+x} \frac{df}{dx} + \varepsilon^2 K f = H$$

and the boundary conditions $f(0, \varepsilon) = f(1, \varepsilon) = 0$, in which K and H are constants. This model has been considered and the first seven terms in the asymptotic series have been found by Fraenkel [4] for a more general case $H = H(x)$. We will extend the series, with H being a constant, by the computer.

As in Model I, setting ε to zero reduces the differential equation to first order. The differential equation suggests $f(x, \varepsilon)$ to be the outer function, x to be the outer variable, and $f(1, \varepsilon) = 0$ to be the only boundary condition being enforced. The governing equation and the outer boundary condition suggest that the outer function has a simple power series expansion as $\varepsilon \to 0$,

$$f(x, \varepsilon) = \sum_{m=0}^{\infty} \varepsilon^m f_{m,0}(x). \tag{11}$$

The solutions for the $f_{m,0}(x)$ are

$$\begin{aligned} m<0 \qquad & f_{m,0}(x) = 0 \\ m=0 \qquad & f_{0,0}(x) = -\frac{H}{2}(1 - \ln x - x) \\ m \geqslant 1 \qquad & f_{m,0}(x) = \int_x^1 dx' \, \frac{1+x}{x} \, (f''_{m-2,0}(x') + K f_{m-2,0}(x')) \end{aligned} \tag{12}$$

The approximation for $f(x, \varepsilon)$ is not valid in the region $x = O(\varepsilon)$. In that region, we introduce the inner function $g(y, \varepsilon)$ with y as the inner variable, again related to the outer function and outer variable as $y = x/\varepsilon$ and $g(y, \varepsilon) = f(x, \varepsilon)$. The inner function $f(y, \varepsilon)$ satisfies the following transformed differential equation, with y fixed and $\varepsilon \to 0$,

$$\frac{d^2 g}{dy^2} + \left(2y \sum_{i=0}^{\infty} (-1)^i \, y^i \varepsilon^i \frac{dg}{dy}\right) + \varepsilon^2 g K = H, \tag{13}$$

and the inner boundary condition $g(0, \varepsilon) = 0$. The logarithmic term in the outer solution suggests that the inner function has the following expansion

$$g(y, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^m \sum_{n=0}^{\mathrm{UILL}(m)} \ln^n \varepsilon g_{m,n}(y), \tag{14}$$

in which UILL(m) is the upper limit in the logarithm that associates with ε^m terms. The value of UILL(m) can be found from the outer expansion solution.[3]

For $m = 0$, UILL $= 1$, $g_{0,1}(y)$ is governed by

$$\frac{d^2 g_{0,1}}{dy^2} + 2y \frac{d g_{0,1}}{dy} = 0,$$

[3] In this case UILL$(m) = (m+2)/2$ for even m and $(m+1)/2$ for odd m.

with the boundary condition $g_{0,1}(0, \varepsilon) = 0$. The solution is

$$g_{0,1}(y) = C_{0,1}\,\mathrm{erf}(y). \tag{15}$$

In general, $g_{m,n}(y)$ is governed by

$$\begin{aligned} m=n=0 \qquad & \frac{d^2 g_{m,n}}{dy^2} + 2y\frac{dg_{m,n}}{dy} = H \\ m>0 \qquad & \frac{d^2 g_{m,n}}{dy^2} + 2y\frac{dg_{m,n}}{dy} = -Kg_{m-2,n} - 2\sum_{i=1}^{m}(-1)^i\, y^{i+1}\frac{dg_{m-i,n}}{dy} \end{aligned} \tag{16}$$

with the boundary condition $g_{m,n}(0) = 0$. We introduce the notation $W_{m,n}(y)$ for the nonhomogeneous terms in Eq. (16), i.e.,

$$\begin{aligned} W_{m,n}(y) &= H && m=n=0, \\ &= -(Kg_{m-2,n}(y) + 2\sum_{i=1}^{m}(-1)^i\, y^{i+1}\frac{dg_{m-i,n}}{dy} && m>0. \end{aligned} \tag{17}$$

The solutions for the $g_{m,n}$ are

$$g_{m,n}(y) = C_{m,n}\,\mathrm{erf}(y) + G2_{m,n}(y) + G3_{m,n}(y),$$

in which

$$\begin{aligned} G2_{m,n}(y) &= \frac{\sqrt{\pi}}{2}\int_0^y dy'\, e^{y'}(1-\mathrm{erf}(y'))\, W_{m,n}(y'), \\ G3_{m,n}(y) &= -\frac{\sqrt{\pi}}{2}(1-\mathrm{erf}(y))\int_0^y dy'\, e^{y'^2} W_{m,n}(y'), \end{aligned} \tag{18}$$

and $C_{m,n}$ are the undetermined constants. The integrands in $G2_{m,n}(y)$ and $G3_{m,n}(y)$ do not have a closed form solution in terms of elementary functions. But fortunately, in order to find the undetermined constants $C_{m,n}$ by matching, only the asymptotic forms for $G2_{m,n}(y)$ and $G3_{m,n}(y)$ as $y \to \infty$ are needed.

3. Programming Experience

We let the computer take over the calculations for the outer and inner functions after the first approximations and then perform the matching for finding the undetermined constants. Our program is run in a strictly interactive mode. In the Appendix, we show the simple algorithm for performing the 0th order matching in the first model.

[4] For the second model, it will be the asymptotic form for the inner function as $y \to \infty$.

Using the algebraic manipulator MACSYMA is generally a lot of fun and often quite rewarding. However, it can also be frustrating. The most serious problem that we encounter is the limited address space in the core. After loading the main MACSYMA package (FIX304 1 DSK MACSYM), we are left 57 blocks of core to do the job. During the execution of programs, these 57 blocks need to accomodate all kinds of data that include compiled functions and arrays, floating point numbers, so-called atomic symbols, uncompiled functions, symbolic expression, etc. In our calculations, the address space is often further reduced by the additional packages (the FASL files) that need to be loaded in for doing a particular manipulation. For example, when a definite integration needs to be done, seven additional package (DIFINT, LIMIT, RESIDU, RPART, SIN, SINIT, SCHATC) will be loaded in and they will take up 40 blocks. Later in the program, if we use the TAYLOR[5] for truncation on a series, MACSYMA will not be able to complete the calculation since the additional FASL file HAYAT for this operation takes up another 17 blocks in the core and leaves us no room for the calculation. In these cases, we often need to find out the number of packages and their sizes for a particular operation and try to avoid large-size operations in the program. As the address space runs low, we save the expressions that we will need and load up a new MACSYMA for further calculation.

It is also frustrating when MACSYMA refuses to do certain manipulation on an expression (usually an extremely complicated one). For example, in some cases, we found that we have to do the truncation before the differentiation, integration, etc. Another time, we found that an infinite recursion is generated when we do the truncation on a product of two infinite series. Later we found out that is caused by one of the infinite series that has only a finite number of non-zero terms. So we introduce "a loop" to cut off the series. We usually find these cures by trial and error.

4. Results and Discussion

MACSYMA has successfully found the first five approximations in the first model. The outer function $f(x, \varepsilon)$ and the inner function $g(y, \varepsilon)$ are

$$
\begin{aligned}
f(x, \varepsilon) = {} & Be^{1-x} - [Be^{1-x}(x-1)] \\
& + \varepsilon^2 Be^{1-x}\left[\frac{x^2}{2} - 3x + \frac{5}{2}\right] \\
& + \varepsilon^3 Be^{1-x}\left[-\frac{x^3}{6} + \frac{15}{6}x^2 - \frac{57}{6}x + \frac{43}{6}\right] \\
& + \varepsilon^4 Be^{1-x}\left[\frac{x^4}{24} - \frac{7}{6}x^3 + \frac{41}{4}x^2 - \frac{187}{6}x + \frac{529}{24}\right] \\
& + \cdots,
\end{aligned}
$$

[5] The usage of a particular function can be found in the MACSYMA Reference Manual [8].

$$
\begin{aligned}
g(y,\varepsilon) = {} & \{Be + e^{-y}(A - eB)\} \\
& + \varepsilon\{Be(1-y) + e^{-y}[y(A - Be) - Be]\} \\
& + \varepsilon^2 \left\{ \left[\frac{Be}{2} y^2 - 2Bey + \frac{5}{2} Be \right] \right. \\
& \left. + e^{-y} \left[-\frac{1}{2}(Be - A)\, y^2 + (A - 2Be)\, y - \frac{5}{2} Be \right] \right\} \\
& + \varepsilon^3 \left\{ \left[-\frac{Be}{6} y^3 + \frac{3}{2} Bey^2 - \frac{11}{2} Bey + \frac{43}{6} Be \right] \right. \\
& + e^{-y} \left[\left(\frac{A}{6} - \frac{Be}{6} \right) y^3 + \left(A - \frac{3}{2} Be \right) y^2 \right. \\
& \left. \left. + \left(2A - \frac{11}{2} \right) y - \frac{43}{6} Be \right] \right\} \\
& + \varepsilon^4 \left\{ \left[\frac{Be}{24} y^4 - \frac{2}{3} Bey^3 + \frac{57}{12} Bey^2 - \frac{50}{3} Bey - \frac{529}{24} Be \right] \right. \\
& + e^{-y} \left[\left(-\frac{Be}{24} + \frac{A}{24} \right) y^4 + \left(-\frac{2}{3} Be + \frac{A}{2} \right) y^3 + \left(-\frac{57}{12} Be + -\frac{5}{2} A \right) y^2 \right. \\
& \left. \left. + \left(-\frac{50}{3} Be + 5A \right) y + \frac{529}{24} Be \right] \right\} \\
& + \cdots .
\end{aligned}
$$

For the second model, MACSYMA found fourteen terms in the series. The calculation for the last four constants $C_{4,3}$, $C_{4,2}$, $C_{4,1}$, $C_{4,0}$ in Eq. (10) was repeated by using the modified matching principle (Eq. 4a). We found that the total time it took for finding the four constants on one was comparable to the time that was used by Fraenkel's matching (Eq. 3) for finding them all at once. In other words, we do not have to use more computer time if we do not need the higher order terms. Also, as the size of the expressions grow in the higher order terms, a MACSYMA will not have enough address space to complete Fraenkel's matching and we will be released from this problem by loading up a new MACSYMA for finding each constant by the modified principle.

The outer and inner functions in the second model are found to be

$$
\begin{aligned}
f(x,\varepsilon) = {} & -\frac{H(-\ln x - x + 1)}{2} \\
& + \varepsilon^2 \left\{ -\frac{HK}{8} \ln^2 x - \frac{HK}{4}(x-1)\ln x + \left[-\frac{HK}{8} x^2 \right. \right. \\
& \left. \left. + \frac{HK}{4} x - \frac{HK - 3H}{4} + \frac{H}{4}\frac{1}{x} - \frac{H}{8}\frac{1}{x^2} \right] \right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\varepsilon^4\left\{\frac{HK^2}{48}\ln^3 x+\frac{HK^2}{16}(x-1)\ln^2 x+\left[\frac{HK^2}{16}x^2\right.\right.\\
&-\frac{HK^2}{8}x+\left(\frac{HK^2}{16}+\frac{3HK}{16}\right)+\frac{HK}{8}\frac{1}{x}+\frac{HK}{16}\frac{1}{x^2}\Big]\ln x\\
&+\left[\frac{HK}{48}x^3-\frac{HK^2}{16}x^2+\left(\frac{HK^2}{16}-\frac{HK}{16}\right)x\right.\\
&-\left(\frac{HK^2}{48}-\frac{5}{8}HK-\frac{41}{96}H\right)-\frac{7}{16}HK\frac{1}{x}-\left(\frac{HK}{8}+\frac{H}{8}\right)\frac{1}{x^2}\\
&\left.\left.-\frac{5}{24}H\frac{1}{x^3}-\frac{3}{32}H\frac{1}{x^4}\right]\right\}\\
&+\cdots.
\end{aligned}
$$

$$
\begin{aligned}
g(y,\varepsilon)\xrightarrow[y\to\infty]{}\;&\ln\varepsilon\left\{\frac{H}{2}\right\}\\
&+\left\{\frac{H}{2}(\ln y-1)+\left(-\frac{H}{8}\frac{1}{y^2}-\frac{3H}{32}\frac{1}{y^4}+\cdots\right)\right\}\\
&+\varepsilon\left\{\frac{H}{2}y-\frac{H}{4}\frac{1}{y}-\frac{5H}{24}\frac{1}{y^3}+\cdots\right\}\\
&+\varepsilon^2\ln^2\varepsilon\left\{-\frac{HK}{8}\right\}\\
&+\varepsilon^2\ln\varepsilon\left\{-\frac{HK}{4}(\ln y-1)+\frac{HK}{16}\frac{1}{y^2}+\cdots\right\}\\
&+\varepsilon^2\left\{\left[-\frac{HK}{8}\ln^2 y+\frac{HK}{4}\ln y-\frac{HK-3H}{8}\right]\right.\\
&\left.+\frac{HK}{y^2}\left[\frac{1}{16}\ln y+\left(-\frac{1}{8}-\frac{1}{8K}\right)\right]+\cdots\right\}\\
&+\varepsilon^3\ln\varepsilon\left\{-\frac{HK}{4}y+\frac{HK}{8}\frac{1}{y}+\cdots\right\}\\
&+\varepsilon^3\left\{-\frac{HK}{4}y(\ln y-1)+\frac{HK}{8y}\left(\ln y-\frac{7}{2}\right)+\cdots\right\}\\
&+\varepsilon^4\ln^3\varepsilon\left\{\frac{HK^2}{48}\right\}\\
&+\varepsilon^4\ln^2\varepsilon\left\{\frac{HK^2}{16}(\ln y-1)+\cdots\right\}
\end{aligned}
$$

$$+\varepsilon^4 \ln \varepsilon \left\{\frac{HK^2}{16}\left(\ln^2 y - 2\ln y + 1 + \frac{3}{K}\right) + \cdots\right\}$$

$$+\varepsilon^4 \left\{-\frac{HK}{8} y^2 + \left[\frac{HK^2}{48}\ln^3 y - \frac{HK^3}{16}\ln^2 y\right.\right.$$

$$\left.\left. + \left(\frac{HK^2}{16} + \frac{3}{16}HK\right)\ln y + \left(-\frac{HK^2}{48} + \frac{5}{8}HK + \frac{41}{96}H\right)\right] + \cdots\right\}$$

$$+\cdots.$$

We would like to give some suggestions for future use of MACSYMA in asymptotic matching:

1. We start letting MACSYMA take over the calculation for the inner and outer functions that are expressed in a closed form (like Eq. (7) and Eq. (10) in Model I) rather than let MACSYMA solve the differential equations that govern the inner and outer functions. The reasons are that the ability in solving differential equations in MACSYMA is still very limited at this moment and having the closed form solution for the inner and outer functions will enable us to know exactly what kinds of operations will be done in the computer.

2. One needs to be cautious when the modified matching principle is used for asymptotic sequences that are mixtures of powers and logarithms in ε. Unless one is absolutely sure about the form of the asymptotic sequence (like the case we have in Model II), one needs to check all the possible switchback terms (Lo [7]). Dealing with the switchback terms is still better than using Fraenkel's matching principle since we will be working on the much simpler lower-order terms instead of those complicated higher-order terms.

3. Extension of series will be limited by storage space or cost. One may want to do the truncation as soon as possible so that the storage will be minimized and the unnecessary cost for handling extra terms will be avoided.

4. One common problem in algebraic manipulation is "intermediate expression swell."[6] We did not encounter this difficulty in our calculations. It will happen most likely in the calculations for the inner and outer functions (the $g_{m,n}(y)$ and $f_{m,n}(x)$). This can be prevented by knowing that we need only the asymptotic series of $g_{m,n}(y)$ as $y \to \infty$ and $f_{m,n}(x)$ as $x \to 0$ in the matching. So one can use the asymptotic forms of all the functions that are involved in the calculations of $g_{m,n}$ and $f_{m,n}$. Since all the operations are done in the series term by term, no intermediate expression swell will be anticipated.

Finally, do not give up when MACSYMA refuses to do a certain manipulation. Most of the time, when we figure out exactly what we want to be done, we can find a way to get around it.

[6] The checking can be done by "STATUS(FREECORE)."

APPENDIX

In this Appendix, we present the simple algorithm for performing the 0th-order matching in the first model. Our program is run in a strictly interactive mode. The (C--) lines are input by the user, while the (D--) lines are the answers from MACSYMA.

We program the basic functions for the first model in the following MACSYMA.

```
:MACSYMA
--Clobber Existing Job-- (Space = yes, Rubout = no)
This is MACSYMA 304
FIX304 1 DSK MACSYM being loaded
Loading done
(C1)  F[M](X) := IF M = 0 THEN B*EXP(1 - X)
         ELSE -EXP(-X)*INTEGRATE (EXP(Z)*
         DIFF (F[M - 1](Z), Z, 2), Z, 1, X)$
(C2)  OF[M](X, E) := IF M = 0 THEN F[0](X)
         ELSE (E^(M))*F[M](X) + OF[M - 1](X, E)$
(C3)  IOF[M, N](Y, E) := SUBST(Y, Y, TAYLOR(OF[M]
         (Y*E, E), E, 0, N))$
(C4)  G[N](Y) := IF N = 0 THEN A + C[0]*(1 - EXP(-Y))
         ELSE C[N]*(1 - EXP(-Y)) - INTEGRATE(G[N - 1](Z), Z, 0, Y)
           + EXP(-Y)*INTEGRATE (G[N - 1](Z)*EXP(Z), Z, 0, Y)$
(C5)  IG[N](Y, E) := IF N = 0 THEN G[0](Y)
         ELSE (E^N)*G[N](Y) + IG[N - 1](Y, E)$
(C6)  OIG[N, M](Y, E) :=
         SUBST (E*Y, X, LIMIT(TAYLOR(IG[N]
         (X/E, E), E, G, M), G, 0, PLUS))$
(C7)  SAVE([LLO1, INNER, DSK, AMES], G, IG, OIG);
(D7)       [[LLO1, INNER, DSK, AMES], 1 BLOCK, G, IG, OIG]
(C8)  SAVE ([LLO1, OUTER, DSK, AMES], F, OF, IOF);
(D8)       [[LLO1, OUTER, DSK, AMES], 1 BLOCK, F, OF, IOF]
(C9)  ^Z
[DDT]
*
```

Equations (7) and (10) are input in lines (C1) and (C4). The functions are saved and we start up a new MACSYMA to perform the 0th order matching.

```
:MACSYMA
--Clobber Existing Job-- (Space = yes, Rubout = no)
This is MACSYMA 304
```

```
FIX304 1 DSK MACSYM being loaded
Loading done
(C1)  LOADFILE(LLO1, INNER, DSK, AMES)$
LLO1 INNERDSK AMES being loaded
Loading done
(C2)  LOADFILE(LLO1, OUTER, DSK, AMES)$
LLO1 OUTER DSK AMES being loaded
Loading done
(C3)  SHOWTIME: TRUE$
Time = 5 msec.
(C4)  M:N:0$
Time = 6 msec.
(C5)  SOLVE(IOF[M, N](Y, E) = OIG[N, M](Y, E), C[N]);
SOLVE FASL DSK MACSYM being loaded
Loading done
HAYAT FASL DSK MACSYM being loaded
Loading done
LIMIT FASL DSK MACSYM being loaded
Loading done
RPART FASL DSK MACSYM being loaded
Loading done
Is X positive, negative, or zero?
POS;
Time = 4125 msec.
```

(D5) $[C_0 = \%E\ B - A]$

```
(C6)  SHOWTIME: FALSE$
(C7)  C[0]: %E*B - A;
```

(D7) $\%E\ B - A$

```
(C8)  OF[0](X, E);
```

(D8) $B\ \%E^{1-X}$

```
(C9)  IG[0](Y, E);
```

(D9) $(\%E\ B - A)(1 - \%E^{-Y}) + A$

```
(C10) STATUS(FREECORE);
(D10)                24 BLOCKS
(C11) SAVE([LLO1, IN0, DSK, AMES], G, IG, OIG);
(D11)      [[LLO1, IN0, DSK, AMES], 1 BLOCK, G, IG, OIG]
(C12) SAVE([LLO1, OUT0, DSK, AMES], F, OF, IOF);
(D12)      [[LLO1, OUT0, DSK, AMES], 1 BLOCK, F, OF, IOF]
(C13) SAVE([LLO1, C, DSK, AMES], C);
```

```
(D13)                [[LLO1, C, DSK, AMES], 1 BLOCK, C]
(C14)  ^Z
[DDT]
*
```

After loading the inner and outer functions in (C1) and (C2), the 0th order matching in Eq. (3) is performed in lines (C4) and (C5). The undetermined constant C_0 is found by MACSYMA in line (D5).

The higher order matching and the matching in the second model follow the same logic. The detail coding can be found in L. Lo [9].

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